Model Answer

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B.Sc. (Hon's) (Fifth Semester) Examination, 2015-16

Mathematics Paper : Second (Real Analysis)

(i) $f_n = 1+\frac{1}{2}+\frac{1}{3}+ - - - - - - - - +\frac{1}{n}$

For $n>m$, we have $|f_n - f_m| = |f_{m+1} + f_{m+2} + - - - - - - + f_n|$

$= \left|\frac{1}{m+1}+\frac{1}{m+2}+ - - - - - - - - - - +\frac{1}{n}\right|$

$= \frac{1}{m+1}+\frac{1}{m+2}+ - - - - - - +\frac{1}{n}$

Take $n=2m$

$|f_{2m}-f_m| = \frac{1}{m+1}+\frac{1}{m+2}+ - - - - - - - - - +\frac{1}{2m}$

$\geq \frac{1}{2m}+\frac{1}{2m}+ - - - - - - - - - - +\frac{1}{2m}$

$= \frac{1}{2m}\times m = \frac{1}{2}$

ie $|f_{2m}-f_m| > \frac{1}{2}$ ——— ①

y Cauchy's general principle of Convergence if $\langle f_n \rangle$ is convergent sequence,
n for each $\epsilon>0$, $\exists$ $m>0$ such that $|f_n - f_m| < \epsilon$ $\forall$ $n \geq m$
If we take $\epsilon = 1/2$ and $n=2m$ then for $\langle f_n \rangle$ to be Convergent sequence, we must $|f_{2m}-f_m| < 1/2$, which contradicts ①.
Hence $f_n = 1+\frac{1}{2}+\frac{1}{3}+ - - - - - +\frac{1}{n}$ cannot converge.

) Sequence $\left\{\sin\frac{n\pi}{3}\right\} = \left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, - - - -\right\}$

Distinct elements of sequence are $\left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$

Limit points of sequence $\left\{\sin\frac{n\pi}{3}\right\}$ will be $\left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$.

ii) $\frac{1}{1\cdot4}+\frac{1}{2\cdot5}+\frac{1}{3\cdot6}+ - - - - - - - - -$

$u_n = \frac{1}{n(n+3)}$, $u_{n+1} = \frac{1}{(n+1)(n+4)}$

By D Alembert Ratio test $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{(n+1)(n+4)}{n(n+3)}$

$= \lim_{n\to\infty}\frac{(1+\frac{1}{n})(1+\frac{4}{n})}{(1+\frac{3}{n})} = 1$

Test fails

Apply Raabe's Test $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(n+1)(n+4)}{n(n+3)}-1\right]$

$= \lim \frac{2n+4}{\quad} = \lim \frac{2+4/n}{\quad} = 2 > 1$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ Given series is Alternating series
Apply Leibnitz's theorem

$u_n = \frac{1}{n^p}$ a) $\because \frac{1}{n^p} \geq 0 \Rightarrow u_n \geq 0$

b) For $p>0$ $(n+1)^p > n^p \Rightarrow \frac{1}{(n+1)^p} < \frac{1}{n^p}$

$\Rightarrow u_{n+1} < u_n \quad \forall n \in \mathbb{N}$

c) $\lim_{n\to\infty} \frac{1}{n^p} = 0 \Rightarrow \lim_{n\to\infty} u_n = 0$

Given alternating series satisfied all 3 conditions of Leibnitz's theorem hence $\sum \frac{(-1)^{n+1}}{n^p}$ is convergent.

Let $P = \{a = x_0, x_1, x_2, ------ x_{r-1}, x_r, x_{r+1}, ---- x_n = b\}$
and $P' = \{a = x_0, x_1, x_2, ----- x_{r-1}, \xi, x_r, ------ x_n = b\}$

$\because P'$ has one more point ξ such that $x_{r-1} < \xi < x_r$ than P, hence P' is refinement of P.

Let

Interval	Supremum	Infremum
$[x_{r-1}, x_r]$	M_r	m_r
$[x_{r-1}, \xi]$	M_r'	m_r'
$[\xi, x_r]$	M_r''	m_r''

Clearly
$M_r \geq M_r'$, $M_r \geq M_r''$
$m_r \leq m_r'$, $m_r \leq m_r''$ — (A)

definition
$$U(P,f) = \sum_{j=1}^{r-1} (x_j - x_{j-1}) M_j + (x_r - x_{r-1}) M_r + \sum_{j=r+1}^{n} (x_j - x_{j-1}) M_j$$
$$U(P',f) = \sum_{j=1}^{r-1} (x_j - x_{j-1}) M_j + (\xi - x_{r-1}) M_r' + (x_r - \xi) M_r'' + \sum_{j=r+1}^{n} (x_j - x_{j-1}) M_j$$
$$\begin{aligned} U(P,f) - U(P',f) &= M_r(x_r - x_{r-1}) - M_r'(\xi - x_{r-1}) - M_r''(x_r - \xi) \\ &= M_r(x_r - \xi) + M_r(\xi - x_{r-1}) - M_r'(\xi - x_{r-1}) - M_r''(x_r - \xi) \\ &= (M_r - M_r'')(x_r - \xi) + (M_r - M_r')(\xi - x_{r-1}) \\ &\geq 0 \quad (\text{From } (A)) \end{aligned}$$

$\Rightarrow U(P,f) \geq U(P',f)$

$$\begin{aligned} U(P,-f) &= \sum_{r=1}^{n} (-m_r)\delta_r \\ &= -\sum_{r=1}^{n} m_r \delta_r \\ &= -L(P,f) \end{aligned}$$

	Sup	Inf
f	M_r	m_r
$-f$	$-m_r$	$-M_r$

and
$$\begin{aligned} L(P,-f) &= \sum_{r=1}^{n} (-M_r)\delta_r \\ &= -\sum_{r=1}^{n} M_r \delta_r \\ &= -U(P,f) \end{aligned}$$

(vii) $\int_0^{\pi} \frac{dx}{\sin x}$

Clearly $x = 0, \pi$ is point of infinite discontinuity

$$= \int_0^{\pi/2} \frac{dx}{\sin x} + \int_{\pi/2}^{\pi} \frac{dx}{\sin x}$$

$$= \lim_{\lambda \to 0+} \int_{0+\lambda}^{\pi/2} \frac{dx}{\sin x} + \lim_{\mu \to 0+} \int_{\pi/2}^{\pi-\mu} \frac{dx}{\sin x} \qquad \left(\begin{array}{l}\text{Conv. at left end at } x=0 \\ \text{Conv. at right " at } x=\pi\end{array}\right)$$

$$= \lim_{\lambda \to 0+} \Big[\log(\operatorname{Cosec} x + \operatorname{Cot} x)\Big]_{\lambda}^{\pi/2} + \lim_{\mu \to 0+} \Big[\log(\operatorname{Cosec} x + \operatorname{Cot} x)\Big]_{\pi/2}^{\pi-\mu}$$

$$= \lim_{\lambda \to 0+} \left\{0 - \log(\operatorname{Cosec} \lambda + \operatorname{Cot} \lambda)\right\} + \lim_{\mu \to 0+} \left\{\log \operatorname{Cosec}(\pi-\mu) + \operatorname{Cot}(\pi-\mu) - 0\right\}$$

$$= \lim_{h \to 0} -\log(\operatorname{Cosec} h + \operatorname{Cot} h) + \lim_{k \to 0} \log\left\{\operatorname{Cosec}(\pi-k) + \operatorname{Cot}(\pi-k)\right\}$$

= does not exist

hence $\int_0^{\pi} \frac{dx}{\sin x}$ is divergent

(viii) $\int_0^{\infty} e^{-x^2} dx$ (Using μ-test) $= \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx$

(Proper)

$$\because \lim_{x \to \infty} x^2 e^{-x^2} = \lim_{x \to \infty} \frac{x^2}{e^{x^2}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{\cancel{2x}}{e^{x^2} \cdot \cancel{2x}} = 0 \quad (\text{finite})$$

Here $\mu = 2 > 1$ Hence by μ-test $\int_1^{\infty} e^{-x^2} dx$ and hence $\int_0^{\infty} e^{-x^2} dx$ will be convergent.

[2] Bounded Sequence: A sequence $\langle S_n \rangle$ is said to be bounded sequence if $\exists$ k, K such that

$$k \le S_n \le K \quad \forall n \in \mathbb{N}$$

Example $S_n = \frac{n}{n+1}$ is bounded sequence for $\frac{1}{2} \le \frac{n}{n+1} \le 1 \quad \forall n \in \mathbb{N}$

We have to show Convergent sequence $\Longleftrightarrow$ bounded + unique limit point

To show Every Convergent sequence is bounded:

Let $\{S_n\}$ be a Convergent sequence converges to l. Then by definition $\forall \epsilon > 0$, $\exists$ $m > 0$ such that $|S_n - l| < \epsilon \quad \forall n \ge m$

$$\Rightarrow l - \epsilon < S_n < l + \epsilon \quad \forall n \ge m$$

Let $k = \min\{S_1, S_2, \dots S_{m-1}, l - \epsilon\}$

$K = \max\{S_1, S_2, \dots S_{m-1}, l + \epsilon\}$

$$\Rightarrow k < S_n < K \quad \forall n \in \mathbb{N}$$

Showing that $\{S_n\}$ is a bounded sequence.

To show every Convergent sequence has a unique limit point:

Let $\{S_n\}$ be a Convergent sequence and if possible suppose $\{S_n\}$ has 2 limit points l and l'.

Now $\lim_{n\to\infty} S_n = l \Rightarrow \forall\, \epsilon > 0,\ \exists\, m_1 > 0 : |S_n - l| < \epsilon/2 \quad \forall\, n \geq m_1$

& $\lim_{n\to\infty} S_n = l' \Rightarrow \forall\, \epsilon > 0,\ \exists\, m_2 > 0 : |S_n - l'| < \epsilon/2 \quad \forall\, n \geq m_2$

Take $m = \max(m_1, m_2)$

$$\Rightarrow \left.\begin{aligned} |S_n - l| &< \epsilon/2 \quad \forall\, n \geq m \\ |S_n - l'| &< \epsilon/2 \quad \forall\, n \geq m \end{aligned}\right\} \text{—— (A)}$$

Consider
$$|l - l'| = |l - S_n + S_n - l'|$$
$$\leq |l - S_n| + |S_n - l'|$$
$$|l - l'| \leq |S_n - l| + |S_n - l'|$$

Using (A)

$$|l - l'| \leq \epsilon/2 + \epsilon/2 \quad \forall\, n \geq m \quad \Rightarrow |l - l'| < \epsilon$$

$\because \epsilon > 0$ is small no. $\Rightarrow |l - l'| = 0 \Rightarrow l = l'$

Hence Convergent sequence $\{S_n\}$ can not have 2 limit points ie every convergent sequence has a unique limit point.

To show a sequence which is bounded and has a unique limit point is convergent:

Let $\{S_n\}$ be a bounded sequence and l is only limit point of $\{S_n\}$. Now any nhbd of l say $(l-\epsilon, l+\epsilon)$ contains infinite number of elements of $\{S_n\}$ and there can be only finite number of elements outside $(l-\epsilon, l+\epsilon)$ for if there are infinite elements outside nhbd $(l-\epsilon, l+\epsilon)$ then there can be another limit point of $\{S_n\}$, but $\{S_n\}$ has unique limit limit. Hence $(l-\epsilon, l+\epsilon)$ contains infinite no. of elements of $\{S_n\}$ except some finite points. Let its number is m

then $S_n \in (l-\epsilon, l+\epsilon) \quad \forall\, n \geq m$

$\Rightarrow |S_n - l| < \epsilon \quad \forall\, n \geq m$

Showing that $\{S_n\}$ converges to l.

3. (a) **Cauchy Criterion for Convergence**

Statement: A sequence converges if and only if it is a Cauchy sequence.

<u>Proof</u>: Let $\{S_n\}$ be a convergent sequence ie $\lim_{n\to\infty} S_n = l$

then by definition $\forall\, \epsilon > 0, \quad \exists\ m > 0$ such that

$$\left.\begin{array}{ll} |S_n - l| < \epsilon/2 & \forall\ n \geq m \\ \text{Take } n = m \quad |S_m - l| < \epsilon/2 & \end{array}\right\} \text{ —— (A)}$$

Now $|S_n - S_m| = |S_n - l + l - S_m|$

$\leq |S_n - l| + |S_m - l|$

Using ①

$|S_n - S_m| < \epsilon \quad \forall\ n \geq m$ showing that $\{S_n\}$ is a Cauchy sequence.

<u>Conversely</u> Let $\{S_n\}$ be a cauchy sequence.

$\because$ Every Cauchy sequence is bounded $\Rightarrow \{S_n\}$ is bounded

and every bounded sequence has limit point $\Rightarrow \{S_n\}$ has limit point l

$\therefore$ nhbd of l ie $(l-\epsilon, l+\epsilon)$ contains infinite elements of $\{S_n\}$.

ie $S_n \in (l-\epsilon, l+\epsilon)$

for $k > m \qquad S_k \in (l - \epsilon/3, l + \epsilon/3) \quad \Rightarrow \quad |S_k - l| < \epsilon/3$ —— (B)

$\because \{S_n\}$ is a Cauchy sequence $\Rightarrow$ for $\epsilon > 0$, $\quad \exists\ m > 0$ such that

$|S_n - S_m| < \epsilon/3 \quad \forall\ n \geq m$ —— (C)

for $k > m \qquad |S_k - S_m| < \epsilon/3$ —— (D)

Consider $|S_n - l| = |S_n - S_m + S_m - S_k + S_k - l|$

$\leq |S_n - S_m| + |S_m - S_k| + |S_k - l|$

Using (B), (C), (D)

$|S_n - l| < \epsilon/3 + \epsilon/3 + \epsilon/3$

$\Rightarrow \quad |S_n - l| < \epsilon \quad \forall\ n \geq m$

Showing that $\{S_n\}$ Converges to l.

(b) $\{f_n\}$ is sequence : $f_n > 0$ and $\lim_{n\to\infty} \dfrac{f_{n+1}}{f_n} = l$

Consider $\{g_n\} = \left\{ f_1, \dfrac{f_2}{f_1}, \dfrac{f_3}{f_2}, \ldots \dfrac{f_n}{f_{n-1}}, \dfrac{f_{n+1}}{f_n}, \ldots \right\}$

Now $g_1 \cdot g_2 \cdots g_n = \cancel{f_1} \cdot \dfrac{\cancel{f_2}}{\cancel{f_1}} \cdot \dfrac{\cancel{f_3}}{\cancel{f_2}} \cdots \dfrac{f_n}{\cancel{f_{n-1}}} = f_n$

$\because \lim_{n\to\infty} \dfrac{f_{n+1}}{f_n} = l \Rightarrow \lim_{n\to\infty} \dfrac{f_n}{f_{n-1}} = l \Rightarrow \lim_{n\to\infty} g_n = l$

$\because f_n > 0 \Rightarrow g_n > 0 \quad \forall\ n \in \mathbb{N}$

By Cauchy second theorem on limit $\because \lim_{n\to\infty} g_n = l$

$\Rightarrow \lim_{n\to\infty} (g_1 \cdot g_2 \cdots g_n)^{1/n} = l \Rightarrow \lim_{n\to\infty} f_n^{1/n} = l$

<u>Proved</u>

[4] Cauchy n^{th} Root test for series:

Statement: Let $\sum u_n$ be a positive term series such that $\lim_{n\to\infty} u_n^{1/n} = l$. Then $\sum u_n$ is

i) Convergent if $l < 1$
ii) divergent if $l > 1$
iii) no firm decision is possible if $l = 1$

Proof: Case-1 Let $l < 1$. take ϵ such that ~~$l < p < 1$~~ $l+\epsilon < 1$

Take $l+\epsilon = \alpha < 1$

$\because \lim_{n\to\infty} u_n^{1/n} = l$ by definition

$\forall \epsilon > 0 \quad \exists m > 0$ such that $|u_n^{1/n} - l| < \epsilon \quad \forall n \geq m$

$\Rightarrow \quad l-\epsilon < u_n^{1/n} < l+\epsilon \quad \forall n \geq m$

$\Rightarrow \quad (l-\epsilon)^n < u_n < (l+\epsilon)^n \quad \forall n \geq m$

$\Rightarrow \quad u_n < \alpha^n \quad \forall n \geq m$ where $\alpha < 1$

Now $\sum \alpha^n$ is a geometric series such that $\alpha < 1$

$\Rightarrow \quad \sum \alpha^n$ is Convergent

By Comparison test of first type $u_n < \alpha^n$

$\because \sum \alpha^n$ is Convergent $\Rightarrow \sum u_n$ is also Convergent.

Case-2 Let $l > 1$ take ϵ such that $l-\epsilon > 1$

say $l-\epsilon = \beta > 1$

$\because \lim_{n\to\infty} u_n^{1/n} = l \quad \forall \epsilon > 0 \quad \exists m > 0$

$|u_n^{1/n} - l| < \epsilon \quad \forall n \geq m$

$l-\epsilon < u_n^{1/n} < l+\epsilon \quad \forall n \geq m$

$(l-\epsilon)^n < u_n < (l+\epsilon)^n \quad \forall n \geq m$

$\Rightarrow \quad u_n > \beta^n \quad \forall n \geq m$ where $\beta > 1$

$\because \sum \beta^n$ is geometric series with $\beta > 1 \Rightarrow \sum \beta^n$ is divergent

by I^{st} Comparison test $u_n > \beta^n$ & $\sum \beta^n$ is divergent

$\Rightarrow \sum u_n$ is divergent.

Case-3 Consider $\sum u_n = \frac{1}{n}$, $\sum v_n = \frac{1}{n^2}$

$\lim_{n\to\infty} u_n^{1/n} = 1$, $\lim_{n\to\infty} v_n^{1/n} = 1$

but $u_n = \sum \frac{1}{n}$ is divergent whereas $v_n = \sum \frac{1}{n^2}$ is Convergent

So in case $l = 1$ no firm decision is possible.

5. (i) $x + x^{1+\frac{1}{2}} + x^{1+\frac{1}{2}+\frac{1}{3}} + x^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} + \cdots$

$$u_n = x^{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}}$$

$$u_{n+1} = x^{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x^{\frac{1}{n+1}}} = 1$$

Hence D Alembert ratio test fails.

Apply Logarithmic test

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n \log \frac{1}{x^{1/n+1}}$$

$$= \lim_{n\to\infty} -\frac{n}{n+1} (\log x)$$

$$= -\log x = \log 1/x$$

If $\log \frac{1}{x} > 1 \Rightarrow x < 1/e \Rightarrow \Sigma u_n$ is convergent.

$\log \frac{1}{x} < 1 \Rightarrow x > 1/e \Rightarrow \Sigma u_n$ is divergent.

$\log \frac{1}{x} = 1 \Rightarrow x = 1/e$ Logarithmic test fails

At $x = \frac{1}{e}$ Apply Higher Logarithmic test

$$\lim_{n\to\infty} \left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n = \lim_{n\to\infty} \left(-\frac{n}{n+1} \log \frac{1}{e} - 1\right) \log n$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1} - 1\right) \log n$$

$$= \lim_{n\to\infty} -\frac{1}{n+1} \log n = \lim_{n\to\infty} -\left(\frac{1}{1+\frac{1}{n}}\right) \frac{\log n}{n}$$

$$= 0 < 1$$

$\Rightarrow \Sigma u_n$ is ~~convergent~~ divergent.

Hence for $\cancel{x \leq \frac{1}{e}}$ $x < \frac{1}{e}$ Σu_n is convergent.

$x \geq \frac{1}{e}$ Σu_n is divergent.

(ii) $\Sigma \sqrt{\frac{3n^2+5n+4}{4n^2+1}}$ $\quad u_n = \sqrt{\frac{3n^2+5n+4}{4n^2+1}} = \sqrt{\frac{3+5/n+4/n^2}{4+1/n^2}}$

$$\lim_{n\to\infty} u_n = \sqrt{\frac{3}{4}} \neq 0$$

Hence by necessary condition for convergence of Σu_n

$\Rightarrow \Sigma u_n$ is divergent.

6. (a) $f(x) = x^3$ defined in $[0, a]$

Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \text{---------} \frac{(r-1)a}{n}, \frac{ra}{n}, \text{----} \frac{na}{n}\right\}$ be partition of $[0,a]$

Now $I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$ $r = 1, 2, \cdots n$ is r^{th} sub-interval.

$\delta_r = \frac{ra}{n} - \frac{(r-1)a}{n} = \frac{a}{n}$ is length of r^{th} sub-interval.

$M_r = \left(\frac{ra}{n}\right)^3 = \frac{r^3a^3}{n^3}$ is supremum of r^{th} " "

$m_r = \left\{\frac{(r-1)a}{n}\right\}^3 = \frac{(r-1)^3a^3}{n^3}$ is infremum of r^{th} " ".

Now $U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} \frac{r^3a^3}{n^3} \cdot \frac{a}{n} = \frac{a^4}{n^4} \sum_{r=1}^{n} r^3$

$= \frac{a^4}{n^4} \cdot \left\{\frac{n(n+1)}{2}\right\}^2 = \frac{a^4}{4}\left(1+\frac{1}{n}\right)^2$

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \frac{(r-1)^3a^3}{n^3} \cdot \frac{a}{n} = \frac{a^4}{n^4} \sum_{r=1}^{n} (r-1)^3$

$= \frac{a^4}{n^4}\left\{\frac{(n-1)n}{2}\right\}^2 = \frac{a^4}{4}\left(1-\frac{1}{n}\right)^2$

$\overline{\int_0^a} f(x)\,dx = \lim_{n\to\infty} U(P,f) = \lim_{n\to\infty} \frac{a^4}{4}\left(1+\frac{1}{n}\right)^2 = \frac{a^4}{4}$

$\underline{\int_0^a} f(x)\,dx = \lim_{n\to\infty} L(P,f) = \lim_{n\to\infty} \frac{a^4}{4}\left(1-\frac{1}{n}\right)^2 = \frac{a^4}{4}$

$\because \overline{\int_0^a} f(x)\,dx = \underline{\int_0^a} f(x)\,dx \Rightarrow f(x)$ is R- integrable in $[0,a]$

$\Rightarrow f \in R[0,a]$

(b) Let f be a monotonic function in $[a,b]$

$\Rightarrow f$ is bounded and let $f(a)$, $f(b)$ be bounds of f and $\epsilon > 0$

let f is monotonically increasing and

$P = \{a = x_0, x_1, x_2, \text{----} x_{r-1}, x_r, \text{----} x_n = b\}$ be partition of $[a,b]$

$I_r = [x_{r-1}, x_r]$ is r^{th} sub-interval, let Length $\delta_r < \frac{\epsilon}{f(b) - f(a) + 1}$

$\delta_r = x_r - x_{r-1}$, $M_r = f(x_r)$, $m_r = f(x_{r-1})$

Now $U(P,f) - L(P,f) = \sum (M_r - m_r)\delta_r$

$= \sum \{f(x_r) - f(x_{r-1})\} \cdot \delta_r$

$< \sum_{r=1}^{n} f(x_r) - f(x_{r-1}) \cdot \frac{\epsilon}{f(b) - f(a) + 1}$

$U(P,f) - L(P,f) < f(b) - f(a) \cdot \frac{\epsilon}{f(b) - f(a) + 1}$

$U(P,f) - L(P,f) < \epsilon$

$\Rightarrow$ function f is R- integrable

hence every monotonic function is integrable.

7. (a) Fundamental Theorem of Integral Calculus :

Statement : If f is bounded, integrable and has primitive ϕ in $[a,b]$ then $\int_a^b f(x)\,dx = \phi(b)-\phi(a)$

Proof: We know that if f is bounded and integrable on $[a,b]$ then $\forall\ \epsilon>0$, $\exists\ \delta>0$ such that $\forall$ Partition P of norm $\leq \delta$ and $\forall\ \xi_r \in [x_{r-1}, x_r]$ we have

$$\left| \sum_{r=1}^{n} f(\xi_r)\,\delta_r - \int_a^b f(x)\,dx \right| < \epsilon \qquad \text{where } \delta_r = x_r - x_{r-1}$$

ϕ is primitive of f $\Rightarrow \phi' = f$ using $f=\phi'$ in above result

$$\left| \sum_{r=1}^{n} \phi'(\xi_r)\,\delta_r - \int_a^b \phi'(x)\,dx \right| < \epsilon \qquad \text{—— (A)}$$

By Lagrange's mean value theorem of differential calculus

$\because$ ϕ is continuous and differentiable in $[x_{r-1}, x_r]$

$\therefore$ $\exists\ \xi_r \in [x_{r-1}, x_r]$ such that

$$\phi(x_r) - \phi(x_{r-1}) = \phi'(\xi_r)(x_r - x_{r-1}) = \phi'(\xi_r)\cdot\delta_r$$

$$\Rightarrow \sum_{r=1}^{n} \phi(x_r) - \phi(x_{r-1}) = \sum_{r=1}^{n} \phi'(\xi_r)\,\delta_r \quad \Rightarrow \quad \phi(b)-\phi(a) = \sum_{r=1}^{n} \phi'(\xi_r)\,\delta_r$$

Put in (A) we get

$$\left| \phi(b)-\phi(a) - \int_a^b \phi'(x)\,dx \right| < \epsilon$$

$\because$ ϵ is small arbitrary positive number

$$\therefore\ \phi(b)-\phi(a) - \int_a^b \phi'(x)\,dx = 0$$

$$\int_a^b \phi'(x)\,dx = \phi(b)-\phi(a)$$

$$\int_a^b f(x)\,dx = \phi(b)-\phi(a)$$

(b) Condition is necessary : Let f be a bounded function on $[a,b]$ and f is integrable over $[a,b]$, then we have

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \qquad \text{—— (1)}$$

From Darboux theorem, f is integrable on $[a,b]$ so $\forall\ \epsilon>0$, $\exists\ \delta>0$ with $\|P\| \leq \delta$ we have

$$\left.\begin{aligned} U(P,f) &< \overline{\int_a^b} f(x)\,dx + \epsilon/2 \\ L(P,f) &> \underline{\int_a^b} f(x)\,dx - \epsilon/2 \end{aligned}\right\}$$

Using (1) in above we get

$$U(P,f) < \int_a^b f(x)\,dx + \epsilon/2 \quad \text{——— (A)}$$

$$L(P,f) > \int_a^b f(x)\,dx - \epsilon/2$$

or $$-L(P,f) < -\int_a^b f(x)\,dx + \epsilon/2 \quad \text{——— (B)}$$

Adding (A) and (B) $\quad U(P,f) - L(P,f) < \epsilon$

Condition is Sufficient $\quad$ We have $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ with $\|P\| \leq \delta$

$$U(P,f) - L(P,f) < \epsilon$$

We know $$\overline{\int_a^b} f(x)\,dx = \inf \{ U(P,f) : P \text{ is partition of } [a,b] \}$$

$$\overline{\int_a^b} f(x)\,dx \leq U(P,f) \quad \text{——— (C)} \quad \text{for any partition } P \text{ of } [a,b]$$

and also $$\underline{\int_a^b} f(x)\,dx = \sup \{ L(P,f) : P \text{ is partition of } [a,b] \}$$

$$\underline{\int_a^b} f(x)\,dx \geq L(P,f) \quad \text{for any partition } P \text{ of } [a,b]$$

$$\Rightarrow \quad -\underline{\int_a^b} f(x)\,dx \leq -L(P,f) \quad \text{——— (D)}$$

Adding (C) & (D) $$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq U(P,f) - L(P,f)$$

but $\quad U(P,f) - L(P,f) < \epsilon$

$$\Rightarrow \quad \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon$$

$\because\ \epsilon$ is small positive number

hence $$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0$$

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx \Rightarrow f \text{ is integrable on } [a,b].$$

[8] $$\int_0^1 (\log 1/x)^m\,dx$$

$x = 0$ and $x = 1$ is point of infinite discontinuity.

$$= \int_0^{1/2} (\log 1/x)^m\,dx + \int_{1/2}^1 (\log 1/x)^m\,dx$$

Convergence at $x = 0$ $\quad \int_0^{1/2} (\log 1/x)^m\,dx$ has $x=0$ is point of infinite discontinuity for $m > 0$

$f(x) = \left(\log 1/x\right)^m$

Take $g(x) = \dfrac{1}{x^p}$, $0 < p < 1$

Now $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} x^p \left(\log 1/x\right)^m \to 0$

$\because \displaystyle\int_0^{1/2} \frac{1}{x^p}\, dx = \int_0^{1/2} g(x)\, dx$ Converges then by Comparison test of type - II $\displaystyle\int_0^{1/2} f(x)\, dx$ ie $\displaystyle\int_0^{1/2} \left(\log 1/x\right)^m dx$ is Convergent $\forall m$.

Convergence at $x = 1$

$$\int_{1/2}^{1} \left(\log 1/x\right)^m dx$$

$x = 1$ is point of infinite discontinuity for $m < 0$

$f(x) = \left(\log 1/x\right)^m$

$g(x) = \dfrac{1}{(1-x)^{-m}}$

$\displaystyle\lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \left(\frac{\log 1/x}{1-x}\right)^m \to 1$

Hence $\displaystyle\int_{1/2}^{1} f(x)\, dx$ and $\displaystyle\int_{1/2}^{1} g(x)\, dx$ behave alike.

$\displaystyle\int_{1/2}^{1} \frac{1}{(1-x)^{-m}}\, dx$ Converges if $-m < 1 \Rightarrow m > -1$

hence $\displaystyle\int_{1/2}^{1} \left(\log 1/x\right)^m dx$ Converges if $-1 < m < 0$

Hence $\displaystyle\int_0^{1} \left(\log 1/x\right)^m dx$ is convergent when $-1 < m < 0$

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